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ON GENERALIZED COUNTABLY APPROXIMATING POSETS

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Abstract. As a generalization of countably approximating posets, we introduce the concept of generalized countably approximating posets. Some properties of generalized countably approximating posets are presented.

1. Introduction

The theory of continuous lattices, due to their strong connections to computer science, general topology and topological algebra have attracted extensive attention (see [3]). One of the important directions in the study of continuous lattices is to carry the theory of continuous lattices to that of posets as much as possible.

In the earlier studies, the notion of continuous posets has proved to be an important generalization of the notion of continuous lattices (see [5]). Another important generalization of continuous lattices is so-called generalized continuous lattices (GCL, for short); they were introduced by Gierz and Lawson in [1, 4] and were called quasicontinuous lattices in [3]. The basic idea is to generalize the way below relation on a complete lattice L to that on the set of subsets of L. As a common generalization of continuous posets and generalized continuous lattices, Gierz, Lawson and Stralka introduced quasicontinuous posets in [2]. Venugopalan studied basic algebraic properties of quasicontinuous posets in [8].

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In 1988, Lee [7] gave the definition of countably way-below relations on a complete lattice, introduced the concept of countably approximating lattices which generalize continuous lattices, and showed that this new larger class enjoyed almost all properties of continuous lattices. In [5], Han, Hong, Lee and Park introduced another class of posets which generalizes countably approximating lattices and continuous posets and studied their properties.

In this paper, we introduce generalized countably approximating posets as a generalization of countably approximating posets by generalizing the countably way below relation on a poset P to that on the set of subsets of P. We prove that the image of a generalized countably approximating poset under an idempotent countably directed join-preserving mapping is a generalized countably approximating poset and give some characterizations of generalized countably approximating posets by the σ_c-topology.

2. Generalized countably approximating posets

Throughout this paper, a partially ordered set is called a poset and P will always mean a poset. A directed complete poset is called a dcpo for short. For a set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite}\}$. A map $f : P \to P$ is called idempotent if $f^2 = f$. For terminology not introduced in the paper, we refer to [3].

Definition 2.1. ([5, 7]) Let P be a poset and D a subset of P.

(1) D is countably directed if every countable subset of D has an upper bound in D.

(2) For any $x,\ y \in P$, we say that x is countably way below y, in symbols $x \ll_c y$ provided for each countably directed subset D of P with $y \leq \bigvee D$, one has $x \leq d$ for some $d \in D$.

(3) A poset with countably directed joins is said to be countably approximating if for all $x \in P$, $\downarrow_c x = \{y \in P : y \ll_c x\}$ is countably directed and $x = \bigvee \downarrow_c x$.

Recall the definition of quasicontinuous posets. Let P be a poset. We say that a nonempty family $\mathcal{F}$ of subsets of P is directed if given $F_1,\ F_2 \in \mathcal{F}$, there exists a non-empty $F \in \mathcal{F}$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$. We define way below relation $\ll$ on the set of subsets of P as follows: $A \ll B$ iff for every directed set $D \subseteq P$, $\bigvee D \in \uparrow B$ implies $D \cap \uparrow A \neq \emptyset$. A dcpo P is called quasicontinuous iff for each $x \in P$, the family $\text{fin}(x)=\{F \in P^{(<\omega)} : F \ll x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F : F \in$

$\mathrm{fin}(x)\}$. A poset P is called a quasicontinuous lattice if it is a complete lattice and a quasicontinuous poset.

Definition 2.2. Let P be a poset. We say a nonempty family $\mathcal{F}$ of subsets of P is countably directed if for any sequence $(F_i)_{i\in\mathbb{Z}_+}$ in $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow F_i$.

Definition 2.3. Let P be poset and $F, G \subseteq P$. We say F is countably way below G and write $F \ll_c G$ if for every countably directed set $D \subseteq P$, $\bigvee D \in \uparrow G$ implies $D \cap \uparrow F \neq \emptyset$. We write $F \ll_c x$ for $F \ll_c \{x\}$.

Definition 2.4. A poset P with countably directed joins is called a generalized countably approximating poset if P satisfies the following conditions:

(1) For each $x \in P$, the family $w(x) = \{F \in P^{(<\omega)} : F \ll_c x\}$ is countably directed;

(2) For each $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$.

The following are immediate and we omit the proof.

Proposition 2.5. *Let P be a poset, then one has:*

(1) *If $F \ll_c G$, then $G \subseteq \uparrow F$.*
(2) *If $F \ll_c G$, $G' \subseteq \uparrow G$, $F \subseteq \uparrow F'$, then $F' \ll_c G'$.*
(3) *If P is a dcpo, then $F \ll G$ implies $F \ll_c G$.*
(4) *If P has countable joins and $(F_i)_{i\in\mathbb{Z}_+}$ is a sequence in $w(x)$, then $F \ll_c x$ and $F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow F_i$, where $F = \{\bigvee_{i\in\mathbb{Z}_+} x_i : x_i \in F_i\}$.*

Lemma 2.6 (Rudin's Lemma). *Let $\mathcal{F}$ be a directed family of nonempty finite subsets of a poset P. Then there exists a directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

The following lemma is crucial in the study of generalized countably approximating posets, and its proof is similar to that of Rudin's Lemma.

Lemma 2.7. *Let $\mathcal{F}$ be a countably directed family of nonempty finite subsets of a poset P. Then there exists a countably directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Using the above lemma, we have the following corollary.

Corollary 2.8. *Let $\mathcal{F}$ be a countably directed family of nonempty finite subsets of a poset P with countably directed joins. If $G \ll_c H$ and $\bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow H$, then $F \subseteq \uparrow G$ for some $F \in \mathcal{F}$.*

LEMMA 2.9. *Let P be a poset and $\mathcal{F}$ a countably directed family of nonempty finite subsets of P. If $f : P \to P$ is a monotone map and $x \in \bigcap_{F\in\mathcal{F}} \uparrow f(F)$, then there exists a countably directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $x \in \bigcap_{d\in D} \uparrow f(d)$ and $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Proof. Consider the collection $\mathcal{A}$ consisting of all $E \subseteq \bigcup_{F\in\mathcal{F}} F$ such that (i) $x \in \bigcap_{e\in E} \uparrow f(e)$; (ii) $E \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, and (iii) $F, G \in \mathcal{F}$ and $G \subseteq\uparrow F$ implies $E \cap G \subseteq\uparrow (E \cap F)$. Clearly $\mathcal{A}$ is not empty, because $B = \{y \in \bigcup_{F\in\mathcal{F}} F : f(y) \leq x\} \in \mathcal{A}$. Order the elements in $\mathcal{A}$ by inclusion. By the Hausdorff Maximality Principle, there exists a maximal $\mathcal{C} \subseteq \mathcal{A}$. Let $D = \bigcap_{C\in\mathcal{C}} C$. Clearly D satisfies (i). That D meets each F follows from the finiteness of F. The finiteness of members of F also yields that D satisfies (iii). Now we prove that there exists $F \in \mathcal{F}$ such that $F \cap D \subseteq\uparrow y$ for all $y \in D$. Suppose that some $y_0 \in D$ has the property $(F \cap D)\setminus \uparrow y_0 \neq \emptyset$ for all $F \in \mathcal{F}$. Then one can verify directly that $D\setminus \uparrow y_0$ again satisfies (i), (ii) and (iii), contradicting the minimality of D. Let $(z_i)_{i\in\mathbb{Z}_+}$ be a sequence in D. Then there exists $F_{x_i} \in \mathcal{F}$ such that $(F_{x_i} \cap D) \subseteq\uparrow x_i$ for all $x \in \mathbb{Z}_+$. Since $\mathcal{F}$ is countably directed, there exists $F \in \mathcal{F}$ such that $F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow F_{x_i}$. For all $i \in \mathbb{Z}_+$, since $F \subseteq\uparrow F_{x_i}$, $D \cap F \subseteq\uparrow (D \cap F_{x_i})$. Then $\emptyset \neq D \cap F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow (F_{x_i} \cap D) \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow x_i$. This shows that D is countably directed and the proof of the lemma is complete. □

LEMMA 2.10. *Let P be a poset with countably directed joins and $\mathcal{F}$ a countably directed family of nonempty finite subsets of P. If $f : P \to P$ preserves countably directed joins, then $\bigcap\{\uparrow f(F) : F \in \mathcal{F}\} =\uparrow f(\bigcap\{\uparrow F : F \in \mathcal{F}\})$.*

Proof. Trivially $\uparrow f(\bigcap\{\uparrow F : F \in \mathcal{F}\}) \subseteq \bigcap\{\uparrow f(F) : F \in \mathcal{F}\}$. Suppose $x \in \bigcap\{\uparrow f(F) : F \in \mathcal{F}\}$. By Lemma 2.9, there exists a countably directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $x \in \bigcap_{d\in D} \uparrow f(d)$ and $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Let $y = \bigvee D$. Then $\bigcap_{d\in D} \uparrow d =\uparrow y$ and $y \in \bigcap_{F\in\mathcal{F}} \uparrow F$. Since f preserves countably directed joins, $f(y) = f(\bigvee D) = \bigvee f(D) \leq x$. Thus $x \in\uparrow f(\bigcap\{\uparrow F : F \in \mathcal{F}\})$. □

LEMMA 2.11. *Let P be a poset with countably directed joins. Then P is a generalized countably approximating poset if and only if for each $x \in P$ there exists a countably directed family $\mathcal{F}$ of nonempty finite subsets of P such that $\mathcal{F} \subseteq w(x)$ and $\bigcap_{F\in\mathcal{F}} \uparrow F =\uparrow x$.*

Proof. We need only prove the sufficiency. Let $\mathcal{F}$ be a countably directed family of nonempty finite subsets of P with $\mathcal{F} \subseteq w(x)$ and $\bigcap_{F\in\mathcal{F}} \uparrow F =\uparrow x$. Then $\uparrow x \subseteq \bigcap_{F\in w(x)} \uparrow F \subseteq \bigcap_{F\in\mathcal{F}} \uparrow F =\uparrow x$.

Thus $\bigcap_{F\in w(x)} \uparrow F = \uparrow x$. Let $(G_i)_{i\in\mathbb{Z}_+}$ be a sequence in $w(x)$. Since $\bigcap_{F\in\mathcal{F}} \uparrow F = \uparrow x$, it follow from Corollary 2.8 that there exists $F_i \in \mathcal{F}$ such that $F_i \subseteq \uparrow G_i$ for all $i \in \mathbb{Z}_+$. Since $\mathcal{F}$ is countably directed, there exists $F \in \mathcal{F}$ such that $F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow F_i$. Then $F \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow G_i$. Since $\mathcal{F} \subseteq w(x)$, $F \in w(x)$. Therefore $w(x)$ is countably directed. □

Remark 2.12. Since $\bigvee \downarrow_c x = x$ if and only if $\uparrow x = \bigcap\{\uparrow y : y \in \downarrow_c x\}$, countably approximating posets are generalized countably approximating posets.

Theorem 2.13. *Let P be a generalized countably approximating poset and $f : P \to P$ be an idempotent self map and f preserves countably directed joins. Then the subposet $f(P)$ is again a generalized countably approximating poset.*

Proof. Let $Q = f(P)$. It is straightforward to verify that Q has countably directed joins and for any countably directed subset D of Q, $\bigvee_P D = \bigvee_Q D$. Let $x \in Q$. Since P is a generalized countably approximating poset, $w_P(x) = \{F \in P^{(<\omega)} : F \ll_c x\}$ is countably directed and $\bigcap_{F\in w_P(x)} \uparrow F = \uparrow x$. Let $\mathcal{F} = \{f(F) : F \in w_P(x)\}$. Then $\mathcal{F}$ is a countably directed family of nonempty finite subset of Q. Now we prove that $\mathcal{F} \subseteq w_Q(x) = \{G \in Q^{(<\omega)} : G \ll_c x\}$. Let D be a countably directed subset of Q with $x \leq \bigvee_Q D = \bigvee_P D$. Then $\uparrow F \cap D \neq \emptyset$ for each $F \in w_P(x)$. Thus $\uparrow f(F) \cap D \neq \emptyset$. Therefore $f(F) \ll_c x$ in Q for all $F \in w_P(x)$.

By Lemma 2.10, $\bigcap_{H\in\mathcal{F}} \uparrow_Q H = \bigcap\{\uparrow_Q f(F) : F \in w_P(x)\} = \uparrow_Q f(\bigcap\{\uparrow_P F : F \in w_P(x)\}) = \uparrow_Q f(\uparrow_P x) = \uparrow_Q f(x) = \uparrow_Q x$. It follows from Lemma 2.11 that $Q = f(P)$ is a generalized countably approximating poset. □

Corollary 2.14. *Let P be a generalized countably approximating poset and if $g : P \to P$ has lower adjoint and g is idempotent, then $g(P)$ is a generalized countably approximating poset.*

Proof. Let h be the lower adjoint of g. Then h is idempotent and h preserves all existing joins. Therefore by Theorem 2.13, $h(P)$ is a generalized countably approximating poset. Since $g(P)$ is order isomorphic to $h(P)$, $g(P)$ is a generalized countably approximating poset. □

Corollary 2.15. *Let P be a generalized countably approximating poset. If $g : P \to Q$ is surjective, preserves countably directed joins and has a lower adjoint. Then Q is a generalized countably approximating poset.*

Proof. Let h be the lower adjoint of g. Then $f = h \circ g$ preserves countably directed joins and $f^2 = f$. By Theorem 2.13, $f(P) = hg(P) = h(Q)$ is a generalized approximating poset. Since $Q = g(P)$ is order isomorphic to $h(Q)$, Q is a generalized countably approximating poset. $\square$

COROLLARY 2.16. *Let P be a generalized countably approximating poset and Q a poset with countably directed joins. If $r : P \to Q$, $s : Q \to P$ preserve countably directed joins and $r \circ s = id_Q$, then Q is a generalized countably approximating poset.*

Proof. Let $f = s \circ r$. Then $f^2 = f$ and f preserves countably directed joins. By Theorem 2.13, $f(P) = s \circ r(P) = s(Q)$ is a generalized countably approximating poset. Since Q and $s(Q)$ are order isomorphic posets, Q is a generalized countably approximating poset. $\square$

3. Topological properties of generalized countably approximating posets

In [5], the authors introduced σ_c-topology on a poset and characterized countably approximating posets by their σ_c-topology. In this section, we discuss the properties of σ_c-topology on generalized countably approximating posets.

Recall the σ_c-topology on a poset P. Let $\sigma_c(P) = \{U \subseteq P : U = \uparrow U$ and for any countably directed subset D of A with $\bigvee D \in U, D \cap U \neq \emptyset\}$. Then $\sigma_c(P)$ is a topology on P, and we call $\sigma_c(P)$ the σ-Scott topology on P.

REMARK 3.1. ([5])(1) For any poset P, the Scott topology $\sigma(P)$ on P is coarser than $\sigma_c(P)$, i.e., $\sigma(P) \subseteq \sigma_c(P)$ and G_δ-sets in $(P,\ \sigma_c(P))$ are open.

(2) A subset C of a poset P is closed in $(P,\ \sigma_c(P))$ iff $C = \downarrow C$ and C is closed under the formation of countably directed joins.

We now consider the interpolation property for generalized countably approximating posets.

PROPOSITION 3.2. *Let P be a generalized countably approximating posets. If $H \ll_c x$, then there exists $F \in P^{(<\omega)}$ such that $H \ll_c F \ll_c x$.*

Proof. Consider the collection $\mathcal{G} = \{G \in P^{(<\omega)}$: there exists $F \in P^{(<\omega)}$ such that $G \ll_c F \ll_c x\}$. Now we prove the following.

(i) $\bigcap_{G\in\mathcal{G}} \uparrow G \subseteq \uparrow x$.

If $z \notin \uparrow x$, then there exists $F \in w(x)$ such that $z \notin \uparrow F$. For each $y \in F$, we can pick $F_y \in w(y)$ such that $z \notin \uparrow F_y$. Set $G = \bigcup_{y \in F} F_y$, then $G \in P^{(<\omega)}$. It is easy to verify that $G \ll_c F$ and $z \notin \uparrow G$.

(ii) $\mathcal{G}$ is countably directed.

Suppose that $G_i \in \mathcal{G}$, $G_i \ll_c F_i \ll_c x$ for $i \in \mathbb{Z}_+$. Since P is generalized countably approximating, there exists $F \in w(x)$ such that $F \subseteq \bigcap_{i \in \mathbb{Z}_+} \uparrow F_i$. Then $G_i \ll_c F$ for all $i \in \mathbb{Z}_+$. Thus $G_i \ll_c y$ for all $y \in F$ and $i \in \mathbb{Z}_+$. For all $x \in F$, $\uparrow y = \bigcap\{\uparrow F' : F' \in w(y)\}$ since P is generalized countably approximating. By Corollary 2.8, there exists $F_i' \in w(y)$ such that $F_y \subseteq \bigcap_{i \in \mathbb{Z}_+} \uparrow F_i' \subseteq \bigcap_{i \in \mathbb{Z}_+} \uparrow G_i$. Set $E = \bigcup_{y \in F} F_y$. Then $E \ll_c F \ll_c x$ and $E \subseteq \bigcap_{i \in \mathbb{Z}_+} \uparrow G_i$.

Since $H \ll_c x$ and $\bigcap_{G \in \mathcal{G}} \uparrow G \subseteq \uparrow x$, by Corollary 2.8, there exists $G \in \mathcal{G}$ such that $G \subseteq \uparrow H$. Since $G \ll_c F \ll_c x$ for some $F \in P^{(<\omega)}$, we conclude that $H \ll_c F \ll_c x$. □

PROPOSITION 3.3. *Let P be a generalized countably approximating poset.*

(i) *For any nonempty set H of P, $\mathrm{int}_{\sigma_c(P)} \uparrow H = \uparrow_c (H)$, where $\uparrow_c (H) = \{x \in P : H \ll_c x\}$.*
(ii) *For any $U \subseteq P$, $U \in \sigma_c(P)$ iff for each $x \in U$ there exists a finite $F \ll_c x$ such that $\uparrow F \subseteq U$. The set $\{\uparrow_c (F) : F \in P^{(<\omega)}\}$ forms a base for $\sigma_c(P)$.*

Proof. (i) By Proposition 3.2, it is easy to verify that $\uparrow_c (H) \in \sigma_c(P)$ by using the interpolation property. Since $\uparrow_c (H) \subseteq \uparrow H$, $\uparrow_c (H) \subseteq \mathrm{int}_{\sigma_c(P)} \uparrow H$. On the other hand, for each $x \in \mathrm{int}_{\sigma_c(P)} \uparrow H$, $\uparrow H \ll_c x$, and then $H \ll_c x$. So $x \in \uparrow_c (H)$. Therefore $\mathrm{int}_{\sigma_c(P)} \uparrow H \subseteq \uparrow_c (H)$.

(ii) Let $U \in \sigma_c(P)$, $x \in U$. From the definition of $\sigma_c(P)$, $U \ll_c x$. So by Proposition 3.2, there exists $F \in P^{(<\omega)}$ such that $U \ll_c F \ll_c x$. Thus $\uparrow F \subseteq U$. Conversely suppose that for each $x \in U$, there exists a finite $F \ll_c x$ such that $\uparrow F \subseteq U$. Then $\uparrow x \subseteq U$, so U is an upper set. Let D be a countably directed set such that $x = \bigvee D \in U$. Then there exists a finite $F \ll_c x = \bigvee D$ such that $\uparrow F \subseteq U$. Thus $U \cap D \neq \emptyset$. From (i), $\uparrow_c (F) \in \sigma(P)$ for all $F \in P^{(<\omega)}$. Thus the set $\{\uparrow_c (F) : F \in P^{(<\omega)}\}$ forms a base for $\sigma_c(P)$. □

Now we give the topological characterizations of generalized countably approximating posets.

THEOREM 3.4. *Let P be a poset with countably directed joins. Then the following conditions are equivalent:*

(1) *P is a generalized countably approximating poset;*
(2) *For all $x \in P$ and $U \in \sigma_c(P)$ with $x \in U$, there exists $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma_c(P)} \uparrow F \subseteq \uparrow F \subseteq U$;*
(3) *$(\sigma_c(P),\ \subseteq)$ is a hypercontinuous lattice.*

Proof. (2) $\Leftrightarrow$ (3) This follows from Lemma 2.2 of [9].

(1) $\Rightarrow$ (2) This follows from Proposition 3.3.

(2) $\Rightarrow$ (1) For all $x \in P$, let $\mathcal{F} = \{F \in P^{(<\omega)} : x \in \mathrm{int}_{\sigma_c(P)} \uparrow F\}$. Since $P \in \sigma_c(P)$, it follows from (2) that there exists $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma_c(P)} \uparrow F \subseteq \uparrow F \subseteq P$. Then $F \in \mathcal{F} \neq \emptyset$. From the definition of $\sigma_c(P)$, $F \ll_c x$ for all $F \in \mathcal{F}$. Then $\mathcal{F} \subseteq w(x)$. Obviously $\uparrow x \subseteq \bigcap_{F\in\mathcal{F}} \uparrow F$. If $x \not\leq y$, then $x \in P \setminus \downarrow y \in \sigma(P) \subseteq \sigma_c(P)$. By (2), there exists $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma_c(P)} \uparrow F \subseteq \uparrow F \subseteq P \setminus \downarrow y$. Then $F \in \mathcal{F}$ and $y \notin \uparrow F$. Thus $\uparrow x = \bigcap_{F\in\mathcal{F}} \uparrow F$. Now we prove that $\mathcal{F}$ is countably directed. Let $(F_i)_{i\in\mathbb{Z}_+}$ be a sequence in $\mathcal{F}$. Since G_δ-sets in $(P,\ \sigma_c(P))$ are open, $x \in \bigcap_{i\in\mathbb{Z}_+} \mathrm{int}_{\sigma_c(P)} \uparrow F_i \in \sigma_c(P)$. Also from (2), there exists $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma_c(P)} \uparrow F \subseteq \uparrow F \subseteq \bigcap_{i\in\mathbb{Z}_+} \mathrm{int}_{\sigma_c(P)} \uparrow F_i$. Therefore $F \in \mathcal{F}$ and $\mathcal{F}$ is countably directed. $\square$

Similarly, we have the following characterizations of countably approximating posets.

THEOREM 3.5. *Let P be a poset with countably directed joins. Then the following conditions are equivalent:*

(1) *P is a countably approximating poset;*
(2) *For all $x \in P$ and $U \in \sigma_c(P)$ with $x \in U$, there exists $y \in P$ such that $x \in \mathrm{int}_{\sigma_c(P)} \uparrow y \subseteq \uparrow y \subseteq U$;*
(3) *$(\sigma_c(P),\ \subseteq)$ is a completely distributive lattice.*

PROPOSITION 3.6. *Let P be a generalized countably approximating poset. If Q is locally closed, i.e., an intersection of an open set and a closed set in $(P,\ \sigma_c(P))$, then with respect to the order inherited from P, Q is a generalized countably approximating poset. Furthermore $\sigma_c(P) \cap Q = \sigma_c(Q)$.*

Proof. First suppose that Q is closed in $(P,\ \sigma_c(P))$. Then since the joins of a countably directed subset of Q will be again in Q, we have that Q is a poset with countably directed joins.

For each $x \in Q$, let $\mathcal{F} = \{F \cap Q : F \in w_P(x)\}$. Remembering Q is closed and hence a lower set, one sees easily that $\mathcal{F}$ satisfies the conditions in Lemma 2.11. Hence Q is a generalized countably approximating poset.

It is easy to prove that $\sigma_c(P) \cap Q \subseteq \sigma_c(Q)$. Conversely let $U \in \sigma_c(Q)$ and $x \in U$. By Theorem 3.4, there exists $F \in Q^{(<\omega)}$, $F \ll_c x$ in Q and $F \subseteq U$. By Corollary 2.8, there exists $G \in P^{(<\omega)}$ such that $G \ll_c x$ of P and $G \cap Q \subseteq \uparrow F$. Again by Theorem 3.4, $\uparrow_c (G) \in \sigma_c(P)$, and we can verify that $x \in \uparrow_c (G) \cap Q \subseteq \uparrow G \cap Q \subseteq \uparrow F \cap Q \subseteq U$.

Now suppose that $Q \in \sigma_c(P)$. Then Q is an upper set and hence a poset with countably directed joins (since P is). For each $x \in Q$, let $\mathcal{F} = \{F \in w_P(x) : F \subseteq Q\}$. By Theorem 3.4, $\mathcal{F} \neq \emptyset$. Then we can check that the collections $\mathcal{F}$ satisfy the conditions in Lemma 2.11. Hence Q is a generalized countably approximating poset. An easy verification yields $\sigma_c(P) \cap Q = \sigma_c(Q)$.

Finally suppose $Q = U \cap C$, where U is open and C is closed in $(P,\ \sigma_c(P))$. Then C is a generalized countably approximating poset by the first part of the proof. Since $U \cap C$ is open in $(C,\ \sigma_c(C))$, then $U \cap C$ is a generalized countably approximating poset by the second part of the proof. □

From the above Proposition, we have the following corollary immediately.

Corollary 3.7. *Let P be a generalized countably approximating poset and Q an open or closed subset in $(P,\ \sigma_c(P))$. Then with respect to the order inherited from P, Q is a generalized countably approximating poset.*

References

[1] G. Gierz and J. D. Lawson, *Generalized continuous lattices and hypercontinuous lattices*, Rocky Mountain J. **11** (1981), 271-296.

[2] G. Gierz, J. D. Lawson, A Stralka, *Quaxicontinuous posets*, Houston J. Mahth. **9** (1981), 271-296.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

[4] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, 1981.

[5] Y. H. Han, S. S. Hong, C. K. Lee, P. U. Park, *A generalization of continuous posets*, Comm. Korean Math. Soc. **4** (1989), no. 1, 129-138.

[6] J. D. Lawson, *The duality of continuous posets*, Houston J. of Math. **5** (1979), 357-394.

[7] S. O. Lee, *On countably approximating lattices*, J. Korean Math. Soc. **25** (1988), no. 1, 11-23.

[8] P. Venugopalan, *Quasicontinuous posets*, Semigroup Forum **41** (1990), 193–200.

[9] X. X. Xu, J. B. Yang, *Topologicl representations of distributive hypercontinuous lattices*, Chinese Ann. of Math. **30**B (2009), 199-206.

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